From Infinite to Finite Belief Contraction

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Abstract

This article contributes to the ongoing discussion on multiple belief change. Based on earlier work, we devise a new condition, called (K$\dot{-}$F), that reduces infinite contraction to sentence contraction. Two representation results reported herein characterize (K$\dot{-}$F) in terms of two well known constructive models for set contraction; namely the partial meet model and the comparative possibility model (a generalization of the epistemic entrenchment model).

Introduction

In the classical framework for *Belief Revision*, now known the *AGM paradigm*, epistemic input is represented as a *single logical sentence* (see (Gardenfors 1988), and (Peppas 2008)). This is the case for both the processes of *belief revision*[1] and *belief contraction* that form the cornerstones of the AGM framework.[2]

Recent work has generalized the AGM paradigm to include epistemic input encoded as a (*possibly infinite*) set of logical sentences, giving rise to the processes of *multiple revision* and *set contraction* (see (Peppas 1996), (Zhang 1996), (Zhang et al. 1997), (Zhang and Foo 2001), (Peppas 2004), (Peppas 2012), and (Peppas, Koutras, and Williams 2012)). There are a number of reasons, other than pure intellectual curiosity, for studing multiple belief change. For example, in a multi-agent setting, merging belief sets is an important process, and so is the revision of a belief set by another belief set. The connection between belief revision and non-monotonic reasoning is another important reason for studying multiple belief change. A third reason is the study of belief revision/contraction in a framework with a rich underlying language, like first-order logic, where one would need to deal with infinite sets of formulas. The reader is referred to (Zhang 1996) and (Zhang and Foo 2001) for further discussion on these issues.

[1]To distinguish the research area from the process, we shall use the capitalized term *Belief Revision* for the former and the same term in lower case letter (i.e. *belief revision*) for the latter.

[2]A third type of belief change that also appears in the AGM framework is *belief expansion*. However, belief expansion is rather simple and clearly not of the same importance as the other two.

Multiple revision and set contraction have been defined axiomatically and constructively. On the axiomatic side, the postulates for multiple revision are straightforward generalizations of the AGM postulates for sentence revision.[3] The case for contraction was more complicated. A slight departure from the spirit of AGM sentence contraction was necessary to accommodate infinite epistemic input. Nevertheless, the postulates for set contraction closely follow their AGM counterparts. On the constructive side, all three major constructive models have been generalized; namely the *system of spheres* model, the *partial meet* model, and the *epistemic entrenchment* model. The generalization of the latter model was also given a new name: the *comparative possibility* model. Finally, representation results have been established connecting the axiomatic and the constructive models.

Once the basic results mentioned above had been established, the attention shifted to the relationship between multiple belief change and sentence belief change. To make the discussion more concrete, let us consider multiple belief revision. Suppose that K is a logical theory, representing the initial belief set of a rational agent, and let Γ be a set of logical sentences representing the epistemic input received by the agent. We are interested in the relationship between the revision of K by Γ, denoted $K * \Gamma$, and the revision of K by the individual sentences in Γ. If Γ is *finite*, then if follows immediately from the postulates that $K*\Gamma = K*(\wedge\Gamma)$ (where $\wedge\Gamma$ denotes the conjunction of all sentences in Γ). If however Γ is infinite, the postulates are not strong enough to reduce multiple revision to sentence revision. Such reductions were proposed independently in (Peppas 1996) and (Zhang et al. 1997), in terms of the conditions (K*F) and (*LP) respectively shown below ($Cn(\Gamma)$ is the logical closure of Γ, and the symbol $\subseteq_f$ stands for "finite subset"; i.e., $A \subseteq_f \Gamma$ states that A is a finite subset of Γ):

(K*F) $\quad K * \Gamma = \bigcap_{A\subseteq_f\Gamma}((K * A) + \Gamma)$

(*LP) $\quad K * \Gamma = \bigcup_{A\subseteq_f\Gamma} \bigcap_{B\subseteq_f Cn(\Gamma)} K * (A \cup B)$

The two conditions are independent of the postulates for multiple revision and provide different means of reducing

[3]By *sentence revision/contraction* we mean classical AGM revision/contraction, where a logical theory K is revised/contracted by a *single logical sentence* φ.

multiple revision to sentence revision.

The relationship between the two conditions, as well as their characterization in terms of systems of spheres, has been studied in (Peppas 2004) and (Peppas 2012).

The counterpart of (*LP) for set contraction is condition ($\dot{-}$LP) below:

($\dot{-}$LP) $\quad K\dot{-}\Gamma = \bigcup_{A\subseteq_f\Gamma}\bigcap_{B\subseteq_f Cn(\Gamma)} K\dot{-}(A\cup B)$

Condition ($\dot{-}$LP) has been mapped into corresponding constraints in the partial meet model (Peppas, Koutras, and Williams 2012), and the comparative possibility model (Peppas 2012).

The same however has not been done for the the set contraction counterpart of (K*F). In the article we fill this last missing piece in the landscape between multiple and sentence belief change. In particular, using generalized versions of the Harper and Levi identities, we firstly formulate the set contraction analog of (K*F), which we call (K$\dot{-}$F). Subsequently, with two representation results, we characterize (K$\dot{-}$F) in terms of constraints for the partial meet model as well as the comparative possibility model.

The article is structured as follows. The next section introduces the necessary definitions and notation, followed by a review on set contraction. Then condition (K$\dot{-}$F) is presented and its relationship with (K*F) is established. Following this, we briefly review the partial meet model and prove our first representation result that characterizes (K$\dot{-}$F) in terms of that model. Finally, after a brief review of the comparative possibility model, we prove our second representation result, that characterizes (K$\dot{-}$F) in terms of comparative possibility. The article ends with some concluding remarks.

Preliminaries

Throughout this paper we shall be working with a formal language L governed by a logic which is identified by its consequence relation $\vdash$. Very little is assumed about L and $\vdash$. In particular, L is taken to be closed under all Boolean connectives, and $\vdash$ has to satisfy the following properties:

(i) $\vdash \varphi$ for all truth-functional tautologies A.

(ii) If $\vdash (\varphi \rightarrow y)$ and $\vdash \varphi$, then $\vdash y$.

(iii) $\vdash$ is consistent, i.e. $\nvdash L$.

(iv) $\vdash$ satisfies the deduction theorem, that is, $\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \vdash y$ iff $\vdash \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n \rightarrow y$.

(v) $\vdash$ is compact.

For a *finite* set of sentences $A = \{\varphi_1, \ldots, \varphi_n\}$, of L we shall use $\wedge A$ to denote the conjunction of all elements of A, i.e. the sentence $\varphi_1 \wedge \cdots \wedge \varphi_n$. For a set of sentences Γ of L, $Cn(\Gamma)$ denotes the set of all logical consequences of Γ, i.e. $Cn(\Gamma) = \{\varphi \in L\colon \Gamma \vdash \varphi\}$. Whenever A is a *finite* subset of Γ, we write $A \subseteq_f \Gamma$.

A theory K of L is any set of sentences of L closed under $\vdash$, i.e. $K = Cn(K)$. We shall denote the set of all theories of L by $\mathcal{K}_L$. A theory K of L is complete iff for all sentences $\varphi \in L$, $\varphi \in K$ or $\neg\varphi \in K$. We shall denote the set of all consistent complete theories of L by $\mathcal{M}_L$. In the context of Belief Revision, consistent complete theories play the role of *possible worlds* and therefore we shall use the two terms interchangeably. For a set of sentences Γ of L, $[\Gamma]$ denotes the set of all consistent complete theories of L that contain Γ. Often we shall use the notation $[\varphi]$ for a sentence $\varphi \in L$, as an abbreviation of $[\{\varphi\}]$. For a theory K and a set of sentences Γ, we shall denote by $K + \Gamma$ the closure under $\vdash$ of $K \cup \Gamma$, i.e. $K + \Gamma = Cn(K \cup \Gamma)$. For a sentence $\varphi \in L$ we shall often write $K + \varphi$ as an abbreviation of $K + \{\varphi\}$. For two sets of sentences Γ, Δ, we define $\Gamma \vdash \Delta$ iff $\Gamma \vdash \delta$ for all $\delta \in \Delta$. Finally, the symbols $\top$ and $\bot$ will be used to denote an arbitrary (but fixed) tautology and contradiction of L respectively.

Set Contraction

Zhang and Foo, (Zhang and Foo 2001), define set contraction as a function $\dot{-} : \mathcal{K}_L \times 2^L \mapsto \mathcal{K}_L$, mapping $\langle K, \Gamma\rangle$ to $K\dot{-}\Gamma$, that satisfies the following postulates:

($K\dot{-}1$) $\quad K\dot{-}\Gamma$ is a theory of L.

($K\dot{-}2$) $\quad K\dot{-}\Gamma \subseteq K$.

($K\dot{-}3$) $\quad$ If Γ is consistent with K then $K\dot{-}\Gamma = K$.

($K\dot{-}4$) $\quad$ If Γ is consistent, then Γ is consistent with $K\dot{-}\Gamma$.

($K\dot{-}5$) $\quad$ If $\varphi \in K$ and $\Gamma \vdash \neg\varphi$ then $K \subseteq (K\dot{-}\Gamma) + \varphi$.

($K\dot{-}6$) $\quad$ If $Cn(\Gamma) = Cn(\Delta)$ then $K\dot{-}\Gamma = K\dot{-}\Delta$.

($K\dot{-}7$) $\quad$ If $\Gamma \subseteq \Delta$ then $K\dot{-}\Delta \subseteq (K\dot{-}\Gamma) + \Delta$.

($K\dot{-}8$) $\quad$ If $\Gamma \subseteq \Delta$ and Δ is consistent with $K\dot{-}\Gamma$, then $K\dot{-}\Gamma \subseteq K\dot{-}\Delta$.

We note that set contraction is different in spirit from AGM sentence contraction. The aim of the former is to contract the initial belief set K in order to make it *consistent* with the epistemic input. On the other hand, the aim of AGM sentence contraction is to reduce K in a way that it *fails to entail* the epistemic input. This shift in the aim of contraction was necessitated by technical considerations that arise when the epistemic input is an infinite set of sentences (see (Zhang 1996) for details).

Multiple revision was also defined by a set of eight postulates, which we shall refer to as $(K*1)$ - $(K*8)$. As already mentioned, $(K*1)$ - $(K*8)$ are straightforward generalizations of the AGM postulates for sentence revision. Due to space limitations, and the secondary role of multiple revision in this article, we will not list the postulates herein (see (Peppas 2008) for details).

We conclude our review of set contraction with the generalized versions of the Levi and Harper Identities:

$$K * \Gamma = (K \dot{-} \Gamma) + \Gamma \qquad \text{(Generalized Levi Identity)}$$

$$K \dot{-} \Gamma = K \cap (K * \Gamma) \qquad \text{(Generalized Harper Identity)}$$

Like in the case of sentence revision/contraction, in (Zhang and Foo 2001) it was shown that any set contraction function satisfying ($K\dot{-}1$)-($K\dot{-}8$), produces through the generalized Levi identity a multiple revision function satisfying (K*1)-(K*8). Conversely, every multiple revision function satisfying (K*1)-(K*8), produces through the generalized Harper identity a set contraction function satisfying ($K\dot{-}1$)-($K\dot{-}8$).

To improve readability, in the rest of this article we shall ignore the limiting cases of contracting (or revising) by an empty or an inconsistent set, and we assume that the epistemic input Γ is always a *non-empty* and *consistent* set of sentences.

From Infinite to Finite Contraction

In view of the generalized Harper identity we can easily devise the set contraction analog of (K*F):

(K$\dot{-}$F) $\quad K\dot{-}\Gamma = \bigcap_{A \subseteq_f \Gamma}((K\dot{-}A) + \Gamma) \cap K$

Condition (K$\dot{-}$F) essentially introduces a method of reducing set contraction by a (possibly *infinite*) set Γ, to a series of contractions by *finite* subsets of Γ. In particular the method consists of three steps. Firstly, the initial belief set K is contracted by each finite subset A of Γ, and then each contracted belief set $K\dot{-}A$ is expanded by Γ. Secondly, all resulting sets $(K\dot{-}A) + \Gamma$ are intersected. Finally, from this intersection we keep only the beliefs that are also in K.

Theorem 1 below proves that (K$\dot{-}$F) is indeed the counterpart of (K*F) for set contraction (due to space limitations, the proof of the theorem is omitted):

Theorem 1 *Let K be a theory, $\dot{-}$ a set contraction function satisfying ($K\dot{-}1$) - ($K\dot{-}8$) and $*$ the multiple revision function generated from $\dot{-}$ via the Generalized Levi Identity. Then $\dot{-}$ satisfies (K$\dot{-}$F) iff * satisfies (K*F).*

The Partial Meet Model

The partial meet model for set contraction, is based on the notion of a *remainder* of a belief set. More precisely, let K be a theory and Γ a nonempty consistent set of sentences. A *remainder* of K with respect to Γ, also called a Γ-*remainder* for short, is any maximal subset of K that is consistent with Γ (Zhang and Foo 2001); the set of all Γ-remainders is denoted by $K \perp\!\!\!\perp \Gamma$. By $\mathcal{R}_K$ we shall denote the set of all remainders of K with respect to any nonempty consistent Γ; i.e. $\mathcal{R}_K = \bigcup\{K \perp\!\!\!\perp \Gamma : \emptyset \neq \Gamma \subseteq L \text{ and } \Gamma \nvdash \bot\}$.

Consider now a preorder $\preccurlyeq$ in $\mathcal{R}_K$. For any nonempty set of remainders $U \subseteq \mathcal{R}_K$, by $max_{\preccurlyeq}(U)$ we shall denote the maximal elements of U with respect to $\preccurlyeq$, i.e. $max_{\preccurlyeq}(U) = \{H \in U : \text{for all } D \in U, D \preccurlyeq H\}$.

A preorder $\preccurlyeq$ on $\mathcal{R}_K$ essentially encodes preference between remainders with the better remainders appearing higher in the preorder. Given this reading, the partial meet model defines the (set) contraction of K by Γ as the theory resulting from the intersection of the best Γ-remainders:

(SC) $\quad K\dot{-}\Gamma = \bigcap max_{\preccurlyeq}(K \perp\!\!\!\perp \Gamma)$

It turns out that the functions induced by (SC) are a superset of those satisfying the postulates for set contraction ($K\dot{-}1$) - ($K\dot{-}8$). To obtain an *exact* match between the two, two extra constraints are needed on $\preccurlyeq$. The first guarantees that the set $max_{\preccurlyeq}(K \perp\!\!\!\perp \Gamma)$ is always well defined:

($\preccurlyeq$ 1) $\quad K \perp\!\!\!\perp \Gamma$ has a maximal element.

For the second constraint we need an extra definition. We define the *closure* of a nonempty set of remainders $U \subseteq \mathcal{R}_K$, denoted $[\![U]\!]$, to be the set $[\![U]\!] = \{H \in \mathcal{R}_K : \bigcap U \subseteq H\}$. If $U = \emptyset$ we define $[\![U]\!] = \emptyset$. We shall say that a set of remainders U is *plain* iff U is equal to its closure; i.e. $U = [\![U]\!]$. The second constraint on $\preccurlyeq$ requires that for all nonempty and consistent Γ, $max_{\preccurlyeq}(K \perp\!\!\!\perp \Gamma)$ is plain:

($\preccurlyeq$ 2) $\quad max_{\preccurlyeq}(K \perp\!\!\!\perp \Gamma) = [\![max_{\preccurlyeq}(K \perp\!\!\!\perp \Gamma)]\!]$.

For a belief set K we shall say that a preorder $\preccurlyeq$ on $\mathcal{R}_K$ is *canonical* iff $\preccurlyeq$ is total, it satisfies ($\preccurlyeq$ 1) - ($\preccurlyeq$ 2), and it has K as its maximal element (i.e. $K' \preccurlyeq K$, for all $K' \in \mathcal{R}_K$).

In (Zhang and Foo 2001), it was shown that contraction functions generated from canonical preorder $\preccurlyeq$ via (SC), coincide precisely with the class of function satisfying the postulates ($K\dot{-}1$) - ($K\dot{-}8$).[4]

Theorem 2 *(Zhang and Foo 2001). Let K be a theory and $\preccurlyeq$ a canonical preorder in $\mathcal{R}_K$. The function $\dot{-}$ defined from $\preccurlyeq$ via (SC) satisfies ($K\dot{-}1$) - ($K\dot{-}8$).*

Theorem 3 *(Zhang and Foo 2001). Let K be a theory and $\dot{-}$ a set contraction function satisfying ($K\dot{-}1$) - ($K\dot{-}8$). There exists a canonical preorder $\preccurlyeq$ in $\mathcal{R}_K$ satisfying (SC).*

We note that there is a close relation between remainders and possible worlds which was proved in (Peppas, Koutras, and Williams 2012) and which will be used extensively in the forthcoming discussion:

Lemma 1 *Let K be a theory. For any remainder $H \in \mathcal{R}_K$ there is a possible world $z \in \mathcal{M}_L$ such that $H = K \cap z$. Conversely, for any $z \in \mathcal{M}_L$, $K \cap z \in \mathcal{R}_K$.*

Two immediate consequences of Lemma 1 are the following useful corollaries:

[4] To be precise, the results in (Zhang and Foo 2001) were stated slightly differently. However it is straightforward to derive Theorems 2 and 3 form the results reported in (Zhang and Foo 2001).

Corollary 1 *Let K be a theory, and H, H' two distinct remainders in $\mathcal{R}_K$. If $H' \neq K$, then $H \not\subseteq H'$.*

Corollary 2 *Let K be a theory, H a remainder in $\mathcal{R}_K$, and Γ a nonempty consistent set of sentences such that $K + \Gamma \vdash \bot$. If Γ is consistent with H, then $H \in K \perp\!\!\!\perp \Gamma$.*

(K$\dot{-}$F) in the Partial Meet Model

To devise the counterpart of (K$\dot{-}$F) in the partial meet model, first we need some extra definitions and notations.

Let K be a theory and $\preccurlyeq$ a preorder in $\mathcal{R}_K$. For any remainder $H \in \mathcal{R}_K$, by $H^{\preccurlyeq}$ we denote the set of all remainders that are greater or equal to H (wrt $\preccurlyeq$); i.e. $H^{\preccurlyeq} = \{D \in \mathcal{R}_K : H \preccurlyeq D\}$. Moreover, for a set of remainders $U \subseteq \mathcal{R}_K$, by $U^{\preccurlyeq}$ we denote the set $U^{\preccurlyeq} = \bigcup_{H \in U} H^{\preccurlyeq}$.

Next consider the condition (PF) below:

(PF) For any nonempty $U \subseteq \mathcal{R}_K$, $[\![U^{\preccurlyeq}]\!] \subseteq U^{\preccurlyeq}$.

Condition (PF) says that for any nonempty set of remainders U, the smallest upper set of $(\mathcal{R}_K, \preccurlyeq)$ that contains U is plain (recall that a set of remainders is plain iff it is identical to its closure).

Although not immediately apparent, (PF) is essentially a *smoothness condition* (similar to the *limit assumption* in the systems of spheres model – see (Peppas 2008)). To see this consider a preorder $\preccurlyeq$ that violates (PF), and let H_0 be a remainder such that $H_0^{\preccurlyeq}$ is not plain. Then there is a remainder $H_1 \in [\![H_0^{\preccurlyeq}]\!]$ that is strictly smaller than H_0 wrt $\preccurlyeq$; i.e. $H_1 \prec H_0$. Likewise, $H_1^{\preccurlyeq}$ may also not be plain, leading to the existence of a $H_2 \in [\![H_1^{\preccurlyeq}]\!]$, such that $H_2 \prec H_1$. Continuing in this fashion, we may eventually end up with an infinitely decreasing chain of remainders $\cdots \prec H_3 \prec H_2 \prec H_1 \prec H_0$, such that for each $i \geq 0$, $H_{i+1} \in [\![H_i^{\preccurlyeq}]\!]$. Hence, if we attempt to find a plain upper set of $(\mathcal{R}_K, \preccurlyeq)$ that contains H_0, we will be moving deeper and deeper into $(\mathcal{R}_K, \preccurlyeq)$ without ever reaching an end. Condition (PF) is design to exclude such anomalies, as shown by Theorem 4 below this is exactly what is needed for reducing infinite contractions to finite contractions:

Theorem 4 *Let K be a theory and $\preccurlyeq$ a canonical preorder in $\mathcal{R}_K$. The set contraction function $\dot{-}$ defined from $\preccurlyeq$ by means of (SC) satisfies (K$\dot{-}$F) iff $\preccurlyeq$ satisfies (PF).*

Proof.

$(\Rightarrow)$

Assume that $\dot{-}$ satisfies (K$\dot{-}$F). Let $U \subseteq \mathcal{R}_K$ be any nonempty subset of $\mathcal{R}_K$. Suppose towards contradiction that $[\![U^{\preccurlyeq}]\!] \not\subseteq U^{\preccurlyeq}$; i.e. there is a $D \in [\![U^{\preccurlyeq}]\!]$ such that $D \notin U^{\preccurlyeq}$.

Clearly $D \neq K$, and therefore by Lemma 1, there is a world $z \in \mathcal{M}_L - [K]$ such that $D = K \cap z$. Consider now the theory $K \dot{-} z$. It is not hard to verify that D is the only element of $K \perp\!\!\!\perp z$ and therefore $K \dot{-} z = D$. As a first step towards the desired contraction we show that for all $A \subseteq_f z$, $K \dot{-} A$ is inconsistent with z.

Consider any $A \subseteq_f z$. Since $D \in [\![U^{\preccurlyeq}]\!]$ we derive that $\bigcap U^{\preccurlyeq} \subseteq D$, and therefore, $\neg(\wedge A) \notin \bigcap U^{\preccurlyeq}$. This again entails that for some $H \in U$, there is a $E \in K \perp\!\!\!\perp A$ such that $H \preccurlyeq E$ and $\neg(\wedge A) \notin E$. On the other hand, from $D \notin U$ we derive that $D \prec H$, for all $H \in U$. Combining the two it follows that $D \notin max(K \perp\!\!\!\perp A)$ and therefore by ($\preccurlyeq$2), there is a $\varphi \in \bigcap max(K \perp\!\!\!\perp A)$ such that $\varphi \notin D$. Since $\bigcap max(K \perp\!\!\!\perp A) = K \dot{-} A \subseteq K$, and $D = K \cap z$, we then derive that $\neg\varphi \in z$ and therefore z is inconsistent with $K \dot{-} A$ as desired.

Since A was chosen arbitrarily, it follows that $\bigcap_{A \subseteq_f z}((K \dot{-} A) + z) = L$ and therefore $(\bigcap_{A \subseteq_f z}((K \dot{-} A) + z)) \cap K = K$. On the other hand, we have shown that $K \dot{-} z = D \neq K$. This clearly violates (K$\dot{-}$F).

$(\Leftarrow)$

Assume that $\preccurlyeq$ satisfies (PF). Let $\Gamma \subseteq L$ be any nonempty, consistent set of sentences. If Γ is consistent with K then so is every finite subset A of Γ and therefore, $K \dot{-} \Gamma = K \dot{-} A = K$, from which (K$\dot{-}$F) trivially follows. Assume therefore that Γ is inconsistent with K.

We proceed in two steps. First we show that for all $A \subseteq_f \Gamma$, if $K \dot{-} A$ is consistent with Γ then $((K \dot{-} A) + \Gamma) \cap K = K \dot{-} \Gamma$. Then we prove that there is at least one such A.

Let A be an arbitrary finite subset of Γ. Clearly, if $K \dot{-} A$ is inconsistent with Γ then clearly $((K \dot{-} A) + \Gamma) \cap K = K$. Consider now the case that $K \dot{-} A$ is consistent with Γ. Then there is at least one Γ-world, call it z, in $[\bigcap max(K \perp\!\!\!\perp A)]$. Let H be the set $H = K \cap z$. By Lemma 1 and Corollary 2, $H \in K \perp\!\!\!\perp \Gamma$. Moreover by construction, $\bigcap max(K \perp\!\!\!\perp A) \subseteq H$ and therefore, by ($\preccurlyeq$2), $H \in max(K \perp\!\!\!\perp A)$. Given that all Γ-remainders are also A-remainders, from the fact that a Γ-remainder (namely H) is among the maximal A-remainders we derive that

$$max(K \perp\!\!\!\perp \Gamma) = \{D \in K \perp\!\!\!\perp A : D \cup \Gamma \nvdash \bot\} \quad (1)$$

Call V the set of worlds, $V = \{z \in \mathcal{M}_L : \text{for some } D \in K \perp\!\!\!\perp A, D = K \cap z\}$. By Lemma 1 and Corollary 2 it follows immediately that $K \perp\!\!\!\perp A = \{K \cap z : z \in V\}$ and therefore $[K \dot{-} A] = [K] \cup [\bigcap V]$. Moreover, from ($\preccurlyeq$2) we derive that $[\bigcap V] = V$.[5] Hence,

$$[K \dot{-} A] = [K] \cup V \quad (2)$$

From (1) and the definition of V we derive that $max(K \perp\!\!\!\perp \Gamma) = \{K \cap z : z \in V \cap [\Gamma]\}$, and consequently, $[K \dot{-} \Gamma] = [K] \cup [\bigcap(V \cap [\Gamma])] = [K] \cup (V \cap [\Gamma]) = ([K] \cup V) \cap ([\Gamma] \cup [K])$.

[5] Clearly $V \subseteq [\bigcap V]$. For the converse, let u be any world in $[\bigcap V]$. Define E to be the set $E = K \cap u$. By Lemma 1, $E \in \mathcal{R}_K$. Moreover by construction, $K \cap (\bigcap V) \subseteq E$, and therefore, $\bigcap max(K \perp\!\!\!\perp A) \subseteq E$. Hence by ($\preccurlyeq$2), $E \in K \perp\!\!\!\perp A$, and therefore $u \in V$ as desired.

Therefore, because of (2), $[K \dot{-} \Gamma] = [K \dot{-} A] \cap ([\Gamma] \cup [K]) = ([K \dot{-} A] \cap [\Gamma]) \cup ([K \dot{-} A] \cap [K])$. Since $[K] \subseteq [K \dot{-} A]$ we then derive that $K \dot{-} \Gamma = ((K \dot{-} A) + \Gamma) \cap K$ as desired.

Hence we have shown that for any $A \subseteq_f \Gamma$, either $((K \dot{-} A) + \Gamma) \cap K = K \dot{-} \Gamma$ or $((K \dot{-} A) + \Gamma) \cap K = K$ depending on whether $K \dot{-} A$ is consistent with Γ. Consequently, either $\bigcap_{A \subseteq_f} ((K \dot{-} A) + \Gamma) \cap K = K \dot{-} \Gamma$, or $\bigcap_{A \subseteq_f} ((K \dot{-} A) + \Gamma) \cap K = K$, depending on whether or not there exists at least one $A \subseteq_f \Gamma$ such that $K \dot{-} A$ is consistent with Γ. We conclude the proof by showing that such an A does indeed exist.

Define U to be the set, $U = \{H \in \mathcal{R}_K :$ there is a $A \subseteq_f \Gamma$ such that $H \in max(K \perp\!\!\!\perp A)\}$. First observe that Γ is consistent with $\bigcap U^{\preccurlyeq}$, for otherwise there is a $B \subseteq_f \Gamma$ such that $\neg(\wedge B) \in \bigcap U^{\preccurlyeq}$, contradicting the fact that, by construction, $K \perp\!\!\!\perp B \subseteq U^{\preccurlyeq}$. Since Γ is consistent with $U^{\preccurlyeq}$, there is a world $z \in [\Gamma]$ such that $z \in [\bigcap U^{\preccurlyeq}]$. Define D to be the set $D = K \cap z$. Clearly, by Lemma 1 and Corollary 2, $D \in K \perp\!\!\!\perp \Gamma$. Moreover by construction $\bigcap U^{\preccurlyeq} \subseteq K$ and $\bigcap U^{\preccurlyeq} \subseteq z$, from which we derive that $\bigcap U^{\preccurlyeq} \subseteq K \cap z = D$. Consequently, $D \in [\![U^{\preccurlyeq}]\!]$ and therefore by (PF), $D \in U^{\preccurlyeq}$. This again entails that all maximal Γ-remainders belong to $U^{\preccurlyeq}$. Hence by the construction of U, there is an $A \subseteq_f \Gamma$ and an $H \in max(K \perp\!\!\!\perp A)$ such that $H \preccurlyeq D$, for all $D \in max(K \perp\!\!\!\perp \Gamma)$. Given that all Γ-remainders are also A-remainders (because $A \subseteq_f \Gamma$), by Corollary 2 we derive that $D \in max(K \perp\!\!\!\perp A)$ and therefore Γ is consistent with $K \dot{-} A$. ■

The Comparative Possibility Model

Apart from the partial meet model, another popular constructive model for (sentence) contraction is *epistemic entrenchment* (Gardenfors and Makinson 1988). As mentioned in the introduction, the epistemic entrenchment model was extended in (Peppas 2012) to cater for infinite epistemic input. The generalized version is called the *comparative possibility model*.

Formally, a *comparative possibility preorder*, relative to a belief set K, is a binary relation $\leq$ between nonempty sets of sentences, that satisfies the following axioms (for all nonempty $\Gamma, \Delta, E \subseteq L$):

(CP1) If $\Gamma \leq \Delta$ and $\Delta \leq E$ then $\Gamma \leq E$.

(CP2) If $\Gamma \vdash \Delta$ then $\Gamma \leq \Delta$.

(CP3) If $\Gamma \nvdash \bot$, then there exists a $z \in [\Gamma]$ such that $\Gamma \leq z$.

(CP4) If $K \nvdash \bot$, then $K \cup \Delta \nvdash \bot$ iff $\Gamma \leq \Delta$ for all nonempty $\Gamma \subseteq L$.

(CP5) If $\Gamma \leq \Delta$ for all nonempty $\Delta \subseteq L$, then $\Gamma \vdash \bot$.

(CP6) If for all $\delta \in Cn(\Delta)$, $\Gamma \leq \Gamma \cup \{\delta\}$, then $\Gamma \leq \Gamma \cup \Delta$.

The intuition behind comparative possibility is different to that for epistemic entrenchment. Rather than encoding the comparative degree of resistance a sentence φ exhibits to its removal, comparative possibility relates to the degree of possibility of (the proposition represented by) a set of sentences: the more possible a set of sentences is, the higher it appears in the comparative possibility preorder (see (Peppas 2012) for a discussion on the origins of comparative possibility).

Given a comparative possibility preorder $\leq$ related to a belief set K, the set contraction of K by a nonempty, consistent set Γ can be constructed by means of the following condition:

(PC) $x \in K \dot{-} \Gamma$ iff $x \in K$ and $\Gamma \cup \{\neg x\} < \Gamma$.

In (Peppas 2012) it was shown that the comparative possibility model is sound and complete with respect to the postulates for set contraction:

Theorem 5 *Let K be a theory and $\leq$ an comparative possibility preorder related to K. Then the function $\dot{-}$ constructed from $\leq$ via (PC) is a set contraction function satisfying ($K\dot{-}1$) - ($K\dot{-}8$).*

Theorem 6 *Let K be a theory and $\dot{-}$ a set contraction function satisfying ($K\dot{-}1$) - ($K\dot{-}8$). Then there exists an comparative possibility preorder $\leq$ related to K which satisfies (PC).*

We conclude our review of comparative possibility with three auxiliary results reported in (Peppas 2012) that we will use in the following section. The first result shows that all comparative possibility preorders are total.

Lemma 2 *Let K be a theory and $\leq$ a comparative possibility preorder related to K. For any two nonempty set of sentences $\Gamma, \Delta \subseteq L$, $\Gamma \leq \Delta$ or $\Delta \leq \Gamma$.*

For the second and third result we need on last definition. For any two nonempty sets Γ, Δ, we define the *set disjunction* of Γ and Δ, denoted $\Gamma \curlyvee \Delta$ as follows:

$$\Gamma \curlyvee \Delta = \{x \vee y : \ \Gamma \vdash x \text{ and } \Delta \vdash y\}$$

Notice that if Γ or Δ is consistent, then $\Gamma \curlyvee \Delta$ is also consistent. Moreover clearly, $\Gamma \vdash \Gamma \curlyvee \Delta$, $\Delta \vdash \Gamma \curlyvee \Delta$, and it is not hard to verify that $[\Gamma \curlyvee \Delta] = [\Gamma] \cup [\Delta]$. The second auxiliary result confines the position of $\Gamma \curlyvee \Delta$ in a comparative possibility preorder:

Lemma 3 *Let K be a theory, $\leq$ a comparative possibility preorder related to K, and Γ, Δ any two nonempty, consistent sets of sentences. If $\Gamma < \Delta$, then $\Gamma \curlyvee \Delta \leq \Delta$.*

The final result form (Peppas 2012) we shall use herein, relates to a condition that is equivalent to (PC):

(SP) $\Gamma \leq \Delta$ iff Δ is consistent with $K \dot{-} (\Gamma \curlyvee \Delta)$ or $\Gamma \vdash \bot$.

Theorem 7 *Let K be a theory, $\leq$ a comparative possibility preorder related to K, and $\dot{-}$ a set contraction function. Then $\dot{-}$ and $\leq$ satisfy (SP) iff $\dot{-}$ and $\leq$ satisfy (PC).*

(K$\dot{-}$F) in the Comparative Possibility Model

Once again we need some extra definitions to formulate the counterpart of (K$\dot{-}$F) in the comparative possibility model.

Let K be a theory and $\leq$ a comparative possibility preorder related to K. For a nonempty set of of possible worlds Y we define $Y^{\leq}$ to be the set $Y^{\leq} = \{z \in \mathcal{M}_L$: there is a $w \in Y$ such that $w \leq z\}$.

Consider now condition (CF) below:

(CF) For any nonempty $Y \subseteq \mathcal{M}_L$, $[\bigcap Y^{\leq}] \subseteq Y^{\leq}$.

There is clearly a strong resemblance between the conditions (CF) and (PF). This is no accident. Like (PF), condition (CF) is essentially a smoothness condition, which turn out to be the analog of (K$\dot{-}$F) in the comparative possibility model:

Theorem 8 *Let K be a theory, $\leq$ a comparative possibility preorder related to K, and $\dot{-}$ the set contraction function induced from $\leq$ via (PC). Then $\dot{-}$ satisfies (K$\dot{-}$F) iff $\leq$ satisfies (CF).*

Sketch of the Proof.

($\Rightarrow$)

Assume that $\dot{-}$ satisfies (K$\dot{-}$F). Let $Y \subseteq \mathcal{M}_L$ be an arbitrary nonempty set of possible worlds. Suppose towards contradiction that there is a $z \in [\bigcap Y^{\leq}]$ such that $z \notin Y^{\leq}$.

Consider now an arbitrary finite subset A of z. From $z \in [\bigcap Y^{\leq}]$ we derive that there is a $w \in Y^{\leq}$ such that $w \vdash A$. Hence from (CP2), $w \leq A$. Moreover from $z \notin Y^{\leq}$ we have that $z < w$. Therefore from (CP1), $z < A$ and consequently from Theorem 7 we derive that z is inconsistent with $K \dot{-} (A \vee z)$. Moreover notice that since $A \subseteq_f z$, it follows that $Cn(A \vee z) = Cn(A)$, and therefore by (K$\dot{-}$6), $K \dot{-} (A \vee z) = K \dot{-} A$. Hence we have shown that for any $A \subseteq_f z$, $(K \dot{-} A) + z = L$, and consequently, $\bigcap_{A \subseteq_f z}((K \dot{-} A) + z) \cap K) = K$.

On the other hand notice that from $z \notin Y^{\leq}$ it follows that z is inconsistent with K and therefore, $K \dot{-} z \neq K$. Hence $K \dot{-} z \neq \bigcap_{A \subseteq_f z}((K \dot{-} A) + z) \cap K)$ which of course contradicts (K$\dot{-}$F).

($\Leftarrow$)

Assume that $\leq$ satisfies (CF) and let Γ be any nonempty consistent set of sentences. Consider any $A \subseteq_f \Gamma$. If $K \dot{-} A$ is inconsistent with Γ then clearly, $((K \dot{-} A) + \Gamma) \cap K = K$. If on the other hand $K \dot{-} A$ is consistent with Γ, it can be shown with the aid of Theorem 7 and (CF), that $((K \dot{-} A) + \Gamma) \cap K = K \dot{-} \Gamma$.

Hence for any $A \subseteq_f \Gamma$, either $((K \dot{-} A) + \Gamma) \cap K = K \dot{-} \Gamma$ or $((K \dot{-} A) + \Gamma) \cap K = K$, depending on whether $K \dot{-} A$ is consistent or not with Γ. Therefore to conclude the proof of (K$\dot{-}$F) it suffices to show that there is at least one $A \subseteq_f \Gamma$ such that $K \dot{-} A$ is consistent with Γ. This can be done with the aid of Theorem 7. ■

Conclusion

This article contributes to the ongoing discussion on multiple belief change. We introduced a new condition, called (K$\dot{-}$F), for reducing infinite contraction to finite contraction. Condition (K$\dot{-}$F) was derived from a similar condition for multiple revision, with the use of the generalized Levi and Harper identities.

The main results of this article are Theorem 4 and Theorem 8 that map (K$\dot{-}$F) into two well known constructive models for set contraction, namely, the partial meet model, and the comparative possibility model respectively.

We note that a condition similar to (K$\dot{-}$F), called the *limit postulate for set contraction*, has also been translated into the partial meet and comparative possibility models (Zhang and Foo 2001), (Peppas 2012). This paper offers an alternative to the limit postulate for set contraction, and the representation results reported herein provide the formal apparatus necessary to evaluate the two different methods of reducing infinite to finite contraction.

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